Are There Toads in Your Class?
Guest Editorial

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Abstract. In teaching our students and preparing them to live in a new and different world from the one we grew up in, we need to think hard about what they need to know and in what ways they need to know it, and then teach it to them in a way that engages them in the exploration of the magical imaginary garden and shows them the importance of the concepts in modern life. Don't allow them out of your class until you have shown them some toads.

The poet, Mariane Moore, was once asked what poetry was about. Her response was that "*Poetry is about imaginary gardens with real toads.*" I have always felt that this wonderful description of poetry serves equally well for mathematics. It describes the duality of theory and application quite well. Mathematics is about imaginary gardens (we make it up) that have real toads (we build bridges that stand through hurricanes, send men to the moon and back).

As mathematicians, we see, even feel, the beauty of mathematics. We enjoy the rich structure of its imaginary gardens. As citizens, we value its many applications in our daily lives. Mathematics is fundamental to the progress we have made during the last century in technology, manufacturing, transportation, medicine and pharmacology, and in every other endeavor in our lives. These toads of mathematics cannot flourish without lots of time playing in the imaginary garden. But equally important, the garden is an empty intellectual game without the toads. In planning for our mathematics classes, we need to consider how much time to spend walking through the imaginary garden and how much time to spend playing with the toads. We need to think carefully and critically about the kinds of toads we show our students and when they should encounter them.

Historically, we have held back the applications of mathematics until after the theory has been thoroughly covered. This has led to a curriculum narrowly defined by a sequence of algebraic manipulations without any serious consideration of why those manipulations are important or in what way they make sense. We have spent all our time on "how to" and little to none on "what for" and "why". And our students have responded appropriately with "who cares?"

Let's consider a standard topic in elementary algebra*: Simplify $\frac{1}{x}+\frac{2}{y}+\frac{3}{z}$. I have told my students with a straight face that when we simplify $\frac{1}{x}+\frac{2}{y}+\frac{3}{z}$ we get $\frac{yz+2xz+3xy}{xyz}$. They look at me, quite appropriately, like I'm crazy. We have successfully changed an expression that involved 5 operations into one that requires 11 operations, and we claim by doing so that we have "simplified" the expression. Just exactly what world are we living in? In what world is $\frac{yz+2xz+3xy}{xyz}$ simpler than $\frac{1}{x}+\frac{2}{y}+\frac{3}{z}$ and why do we care? Where is the toad in this kind of rewriting of rational expressions?

One of the essential features of our form of government is the reapportionment of the 435 members of the US House of Representatives every 10 years. To understand the fairness of our past and present methods of apportionment and to appreciate the mathematical capabilities of our founding fathers, students need to be reasonably facile with simplification such as that shown above. Once they have practiced simplifying rational expressions students can see how these expressions can and have altered history.

That's just one toad, but a very important one for citizens of the United States (see *Apportionment: Measuring Unfairness* in **Consortium**, Number 81, Spring, 2002. for details).

As a second example, consider the number $\frac{10}{\sqrt{99}}$. A little thought will tell the students that the value is a little larger than 1, since $\sqrt{99}$ is a little less than 10. Now, suppose we ask a student to "simplify" $\frac{10}{\sqrt{99}}$. What do we expect them to do? Again with a straight face, we have them rewrite $\frac{10}{\sqrt{99}} = \frac{10}{3\sqrt{11}}\left(\frac{\sqrt{11}}{\sqrt{11}}\right) = \frac{10\sqrt{11}}{33}$. Now, give me a quick estimate of the size of $\frac{10\sqrt{11}}{33}$? How can we take a perfectly good irrational number like $\frac{10}{\sqrt{99}}$ and mash it all out of shape until we get $\frac{10\sqrt{11}}{33}$, and claim that we now have a "simpler" representation? In what world is $\frac{10\sqrt{11}}{33}$ simpler than $\frac{10}{\sqrt{99}}$?

At one time, we knew why we did all this. Here is a page from *Ray's Algebra*, published in 1866. Notice the remark in the middle of the page.

"The object of the above [rationalizing denominators] is to diminish the amount of calculation in obtaining the numerical value of a fractional radical. Thus, suppose it is required to obtain the numerical value of the fraction $\frac{\sqrt{2}}{\sqrt{3}}$ in example 2 above, true to six places of decimals. Here, we may first extract the square root of 2 and of 3 to seven places of decimals, and then divide the first result by the second. This operation is very tedious. If we render the denominator rational, the calculation consists in finding the square root of 6, and then dividing by 3".

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Reduce the following fractions to equivalent fractions, having rational denominators:

1. $\frac{1}{\sqrt{2}}$. . . . . Ans. $\frac{\sqrt{2}}{2}=\frac{1}{2}\sqrt{2}$.
2. $\frac{\sqrt{2}}{\sqrt{3}}$. . . . . Ans. $\frac{\sqrt{6}}{3}=\frac{1}{3}\sqrt{6}$.
3. $\frac{3}{6-\sqrt{3}}$. . . . . Ans. $\frac{1}{11}(6+\sqrt{3})$.
4. $\frac{5}{\sqrt{7}+\sqrt{6}}$. . . . . Ans. $5(\sqrt{7}-\sqrt{6})$.

REMARK.—The object of the above is to diminish the amount of calculation in obtaining the numerical value of a fractional radical. Thus, suppose it is required to obtain the numerical value of the fraction $\frac{\sqrt{2}}{\sqrt{3}}$ in example 2 above, true to six places of decimals.

Here, we may first extract the square root of 2 and of 3 to seven places of decimals, and then divide the first result by the second. This operation is very tedious. If we render the denominator rational, the calculation merely consists in finding the square root of 6, and then dividing by 3.

5. Find the numerical value of $\frac{3}{\sqrt{5}}$. Ans. 1.3416407+.
6. Of $\frac{\sqrt{2}}{\sqrt{5}-\sqrt{3}}$. . . . . . Ans. 2.805883+.

Evidently, to save paper in future additions of algebra texts, this explanation for why we rationalize denominators was left out, but the example problems remained. And they remain to this day.

But this note from 1866 allows us to see clearly the world in which these expressions are indeed simplified. Imagine a world in which addition and subtraction are essentially free. Multiplication will cost you a little. But division is very expensive. And division by "bad numbers", you can't afford. In such a world, we have simplified our expressions by trading three very expensive division operations in $\frac{1}{x}+\frac{2}{y}+\frac{3}{z}$ for ten cheap operations and one expensive division in $\frac{yz+2xz+3xy}{xyz}$ and trading a division we cannot afford in $\frac{10}{\sqrt{99}}$ for the expensive, but still achievable division in $\frac{10\sqrt{11}}{33}$. Almost everything we do in elementary algebra that goes by the name of “simplify” follows from this computational cost accounting.

Does this mean that students shouldn’t learn how to simplify rational and radical expressions? Of course not, but it certainly means that this algebraic manipulation is not all that they should learn. It also means that we as teachers need to have some reasonable perspective on how much time to spend in this imaginary garden of symbol manipulation. Students should know that by learning to manipulate algebraic expressions, they will eventually be able to solve more interesting and more important problems (these manipulations are not an end, but a means). And students shouldn’t leave the course without actually seeing some of those problems, and playing with the toads of elementary mathematics.

In teaching our students and preparing them to live in a new and different world from the one we grew up in, we need to think hard about what they need to know and in what ways they need to know it, and then teach it to them in a way that engages them in the exploration of the magical imaginary garden and shows them the importance of the concepts in modern life. Don’t allow them out of your class until you have shown them some toads.

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*I'd like to thank Henry Pollak of Teacher's College at Columbia for this example.