Continued Fractions as a Professional Development Tool

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Abstract: This manuscript is a continuation of the description of a project that offered middle-school mathematics teachers the opportunity to participate in mentored research in mathematics. We look at the study of continued fractions both as a learning experience for the participant and as a context for developing student oriented teaching methods.

The Departments of Mathematics from Southeastern Louisiana University and Louisiana State University partnered to provide professional development to middle school mathematics teachers. The uniqueness of the program was offering the teachers an opportunity to conduct mentored research in mathematics. The protocol used was to determine, with the mentees, mathematical topics of interest to them or to their students. Time was spent answering questions related to the topic of interest, and as a final product, the mentees prepared and carried out a plan to teach appropriate parts of the research to their students. The purpose of this manuscript is to describe such a mentored research opportunity with a middle school faculty teaching at grades 7 and 8.

Selecting the Research Question:

"What topic can we choose that affects many students in junior high mathematics?" "What do we want to study that could be related to junior high mathematics?" "Geometry?" "Topology?" "Fractions?" "Continued Fractions!" "What is a continued fraction?" "What is a role for continued fractions in mathematics?"

The questions above were the start of one of the neatest mathematical experiences that I have had. As indicated, the context for these questions was a summer research experience for middle school teachers. I had been fortunate to have as my "mentee" a junior high mathematics teacher who had completed an undergraduate mathematics major. As the questions above indicate, the idea of her researching continued fractions just worked itself out in conversation; and it was an excellent choice of topic that led to several research questions.

Researching the question:

Our research process was a series of one-on-one discussions with the "mentee" discussing material on continued fractions that she had read. The questions researched were the following:

Question 1: "What is a continued fraction?"

Question 2: "What is a role for continued fractions in mathematics and in the teaching of mathematics?"

In order to understand the mathematics of continued fractions, it was necessary that the mentee spend considerable time going from a rational number to the continued fraction representation of the rational number. These computations led to our posing various conjectures about continued fractions.

To begin, the mentee looked at the continued fraction representation of 119/35.

$$\frac{119}{35}=3+\frac{14}{35}=3+\frac{1}{\frac{35}{14}}=3+\frac{1}{2+\frac{7}{14}}=3+\frac{1}{2+\frac{1}{\frac{14}{7}}}=3+\frac{1}{2+\frac{1}{2+\frac{0}{7}}}=<3;2,2>.$$

The notation $<d_1;d_2,d_3,d_4,...>$ is defined as $d_1+\cfrac{1}{d_2+\cfrac{1}{d_3+\cfrac{1}{d_4+...}}}$.

The relevance to junior high mathematics began to show itself so well: in every step, we were inverting fractions and rewriting an "improper" fraction as a whole number plus a fractional part. Could we get junior high students to be as excited about working with fractions as we were?

Note that our expansion of 119/35 terminated. A 0 or 1 (such as 0/7) signifies the end of the process.   We immediately had another question:

Question 3: "Do all continued fraction expansions of "numbers" terminate?"

Theorem: A continued fraction expansion of a number terminates if and only if the given number is a rational number.

Pedagogical question: Is this a theorem that junior high students could deduce from working several examples? This question would later be incorporated into the teacher's lesson.

Question 4: What is the process of changing a continued fraction expansion such as <1; 2,1,3> to a rational number?

Answer: $$1+\cfrac{1}{2+\cfrac{1}{1+\frac{1}{3}}}=1+\cfrac{1}{2+\cfrac{1}{\frac{4}{3}}}=1+\cfrac{1}{2+\frac{3}{4}}=1+\cfrac{1}{\frac{11}{4}}=1+\frac{4}{11}=\frac{15}{11}$$

Again, the facility to work with fractions becomes an end-product of exercises such as this one.

Question 5: Our next question involved the continued fraction expansion of reciprocals. The "mentee" did several examples, read material and concluded the following:

Let a and b be positive integers with a > b. If $\frac{\mathrm{a}}{\mathrm{b}} =< d_1; d_2, d_3, ..., d_n >$,

then $\frac{\mathrm{b}}{\mathrm{a}} =< 0; d_1, d_2, d_3, ..., d_n >$.

Try this with 35/119 as a nice example!

Question 6: How does the continued fraction representation of a fraction that is not in lowest terms compare to the continued fraction representation of the fraction in simplest terms?

At this point, several references suggested that we take a look at the "convergents" of a continued fraction. We researched the topic of convergents and applied this knowledge to several rational numbers.

Question 7: What are the convergents of a continued fraction?

The convergents of a continued fraction are obtained by terminating the continued fraction expansion after each step in the expansion. That is, if $r = < d_1; d_2, d_3, ..., d_n >$ is the continued fraction representation of the rational number, r, then the convergents of r are $d_1; d_1 + \frac{1}{d_2}; d_1 + \cfrac{1}{d_2 + \cfrac{1}{d_3}};$ and so forth. For example, the convergents of

$$\frac{43}{30} = <1; 2, 3, 4> \text{ are}$$

$$1 = 1;\ \frac{3}{2} = 1.5;\ \frac{10}{7} = 1.42871 \text{ (repeating)};\ \frac{43}{30} = 1.4333...$$

Question 8: What about the convergents of non-terminating continued fractions? We had been working with terminating continued fractions; now, we were looking at non-terminating continued fractions. The definition of convergents is the same for non-terminating continued fractions as for terminating continued fractions. The convergents of π seem to be a common approach to learning about convergents in the context of non-terminating continued fractions.

If we use 3.14159 as an approximation of π, the continued fraction representation for this value can be computed as follows:

$$3.14159 = 3 + \frac{14159}{100000} = 3+\frac{1}{\frac{100000}{14159}} = 3+\frac{1}{7+\frac{887}{14159}} = 3+\frac{1}{7+\frac{1}{\frac{14159}{887}}} =$$

$$3+\frac{1}{7+\frac{1}{15+\frac{854}{887}}} = 3+\frac{1}{7+\frac{1}{15+\frac{1}{\frac{887}{854}}}} = 3+\frac{1}{7+\frac{1}{15+\frac{1}{1+\frac{33}{854}}}} =$$

$$3+\frac{1}{7+\frac{1}{15+\frac{1}{1+\frac{1}{\frac{854}{33}}}}} = 3+\frac{1}{7+\frac{1}{15+\frac{1}{1+\frac{1}{25+\frac{29}{33}}}}} \ \ldots$$

The convergents of $3+\frac{1}{7+\frac{1}{15+\frac{1}{1+\frac{1}{25+\frac{29}{33}}}}}$ are

$(\frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{9208}{2931}, \ldots)$. Computing the convergents of several irrational numbers helped to understand the patterns that were formed.

This discussion of convergents led to two excellent research topics: the use of the Euclidean Algorithm to find the convergents of a continued fraction, and the concept of the convergence of a sequence of numbers. We chose to investigate the use of the

Euclidean Algorithm within this topic. Convergence of a sequence of real numbers could be our summer experience next summer!!

Question 9: How can we use the Euclidean Algorithm to deduce an iterative process to find the convergents of a continued fraction.

Take a look at the convergents of $\frac{43}{30}$. Let $\frac{p_i}{q_i}$, $i = 1,\ 2,\ 3,\ \ldots$ be the i^{th} convergent of a continued fraction. Note that $p_i = d_i p_{i-1} + p_{i-2}$ and $\mathrm{q}_{\mathrm{i}} = d_i q_{i-1} + q_{i-2}, i = 3, 4, 5, \ldots$ with $p_1 = d_1,\ q_1 = 1,\ p_2 = d_2 p_1 + 1,\ and\ q_2 = d_2$. We use these relationships among the p_i's and the q_i's to complete the following table:

i	1	2	3	4
d_i	1	2	3	4
p_i	1	3	10	43
q_i	1	2	7	30

In a similar fashion, we can find the convergents of π:

i	1	2	3	4	5
d_i	3	7	15	1	25
p_i	3	22	333	355	9208
q_i	1	7	106	113	2931

Question 10: Prompted by the discussion of convergents, we asked, "How do you use the decimal expansion of a real number to locate its position on the real line?" (Our answer was similar to working with a nested sequence of closed intervals on the real number line.)

We then looked at the placement of π on the real number line and considered the convergence of a sequence of real numbers without explicitly defining "convergence of a sequence." This should lead to an excellent segue into epsilon-delta conversations.

Question 11: Another question that came up in our discussions: How can we calculate the value of π?

One answer: Arctan 1 = 1 – 1/3 + 1/5 – 1/7 + …

Arctan 1 = π/4. Thus, π = 4(1 – 1/3 + 1/5 – 1/7 + …).

Analyzing the research results:

There are two levels on which we can analyze the results of this mathematical research experience. One is the mathematics that is newly learned by the teacher-participant, and the other is any change in the teaching and professional manners of the teacher-participant.

On the level of learning new mathematics, the results were positive. Certainly, the topic of continued fractions provided the mentee with several new mathematical learning opportunities and extended her mathematics into areas not usually found in the junior high mathematics curriculum.

Probably an even more exciting result of this mathematical experience was the change in the mentee's approach to teaching mathematics. During the experience, she spent one week away with her family. During this time, she attempted to re-write most of her junior high mathematics curricular materials into a problem-solving format similar to the questions about continued fractions that we had asked of ourselves. She revised her lessons into questions and problems that she could pose to her students to get them to do mathematics. This was a momentous result that showed itself in the evaluation of the summer experience.

Incorporating the newly gained knowledge appropriately:

The following is an outline of the lesson that was developed:

1. Expand 49/38 into a continued fraction.
2. Introduce <a; b, c, d…> notation.
3. Have students expand 11/3 into a continued fraction.
4. Other examples: 42/11; 8/5.
5. Expand 1.7 into a continued fraction.
6. Expand 2.6 into a continued fraction. (Leave this example on the board.)
7. Compare continued fraction expansion of 2.6 with continued fraction expansion of 2 3/5.
8. How would we compute the continued fraction expansion of a fraction with value between 0 and 1?
9. Go to groups: How does any continued fraction representation compare with the continued fraction representation of its reciprocal?
10. Still in groups: How will the continued fraction representation of a fraction that is not in lowest terms compare to the continued fraction representation of the fraction in simplified form?
11. Find the convergents of 43/10; define convergents from this example.
12. Work with the students to develop the generalizations for p_i *and* q_i.
13. Work with the students to use these generalizations to find convergents for π

NOTE: For junior high students, #12 and #13 become a task in learning how to read subscripts and an indexed array of numbers.

Presenting the Lesson:

Two different lesson plans were prepared: one higher level lesson plan and one lower level lesson plan. The one to be used would depend on the algebraic level of the junior high students who would participate. The higher level lesson plan was used because the students were considered more advanced..

The students were shown how to expand 49/38 and 43/30 into their respective continued fraction representation. With only these two examples as practice, the students were able to do such work on their own. They expanded 11/3; 42/11; 1.7; and 2.6. They were quick to do the arithmetic, and they were accurate in their computations. The students compared the continued fraction representation of 4/7 with that of 7/4 and the continued fraction representation of 43/30 with that of 30/43.

The discussion then went to the convergents of a continued fraction. This was probably too much of a challenge for the students, but given sufficient days to develop these concepts at a slower pace, convergents should be a good topic for the higher level students.

Summary

As indicated at the start of the article, this experience was one of the most rewarding mathematical experiences that I have had. We are analyzing the success of each of the individual groups of mentees with the plan of carrying this project to other middle-school teachers.

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